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Cylindrically Symmetric Zel'dovich Fluid Distributions in General Theory of Relativity

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The problem of charged perfect fluid distribution is investigated when the space-time is described by the Einstein–Rosen metric. It is shown that with assumed cylindrical symmetry the cosmological constant Λ vanishes, the electromagnetic field becomes source-free, and the perfect fluid reduces to Zel'dovich fluid with $p = \rho$. Sets of exact solutions for this case have been obtained and the corresponding solutions for Brans–Dicke–Maxwell fields have been derived. For these solutions the Einstein–Rosen metric, however, goes over to three-parameter Marder metric in Brans–Dicke theory.

1. INTRODUCTION

Misra and Radhakrishna (1962) have obtained sets of exact solutions for the case of cylindrically symmetric source-free electromagnetic fields when the space-time is described by the Einstein–Rosen metric. Considering the same metric, Rao et al. (1972, 1973) have obtained solutions for coupled zero-mass scalar and source-free electromagnetic fields and have studied the physical behavior of those solutions. They (Rao et al., 1974a, b, 1975, 1978) have also extended the study to the case of Brans–Dicke fields.

Recently Rao et al. (1980a, b) have tried to consider the problem of a charged perfect fluid in cylindrically symmetric space-time described by the Einstein–Rosen metric. However, owing to the symmetry imposed by the metric the problem has reduced to the case of interacting source-free electromagnetic field and Zel'dovich (1962, 1972) fluid (characterized by equivalency of pressure and density) distributions. In this paper they (Rao et al., 1980a, b) have obtained a class of solutions and have shown how to generate source-free electromagnetic and Zel'dovich fluid solutions from those of Zel'dovich fluid solutions. However, this class of solutions does not

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exhaust all possible solutions. The present work is an extension of the work of Rao et al. (1980a, b). We have considered here, as before, the problem of Rao et al. (1980a, b) (i.e., charged perfect fluid) with cosmological constant Λ, where we assume Λ to be positive in order to have gravity theory as an analog of particle physics (Wesson, 1980). The symmetry condition again forces the electromagnetic field to becomes source-free, the perfect fluid reduces to a Zel'dovich fluid with equation of state $p = \rho$, and the cosmological constant Λ vanishes. Sets of exact solutions for this case have been obtained.

Since the Zel'dovich fluid behaves as a zero-mass scalar field when the fluid is suitably restricted, using the transformation of Tabensky and Taub (1973) we have generated the corresponding solutions of our problem in the Brans–Dicke theory (henceforth referred to as the BD theory). It turns out, however, that the two-parameter Einstein–Rosen metric goes over to the three-parameters Marder (1958a, b) metric. Incidentally we may remark that the electromagnetic field remains unchanged owing to the conformal properties of the Tabensky and Taub (1973) transformation.

2. FIELD EQUATIONS

Einstein field equations for the region of space-time in the presence of a charged perfect fluid distribution are

$$G_{ij} \equiv R_{ij} - \tfrac{1}{2} g_{ij} R + \Lambda g_{ij} = -K(T_{ij} + T'_{ij}) \tag{1}$$

where K $(=8\pi G, C = 1)$ is the gravitational constant, and

$$T_{ij} = \frac{1}{4\pi}\left(-F_{is}F_j^s + \frac{1}{4} g_{ij} F_{ab} F^{ab} \right) \tag{1a}$$

and

$$T'_{ij} = \frac{1}{4\pi}\left[(\rho + p) U_i U_j - p g_{ij} \right], \qquad U_i U^i = 1 \tag{1b}$$

are electromagnetic and perfect fluid stress energy tensors, respectively. F_{ij}, the electromagnetic field tensor, satisfies the field equations

$$F_{ij} = A_{i,j} - A_{j,i} \tag{2}$$

$$F^{ij}_{\ ;j} = -4\pi U^i \sigma \tag{3}$$

where A_i and U^i are the electromagnetic four-potential and four-velocity vectors, respectively. Here ρ, p, and σ are, respectively, mass density, pressure, and charge density. A comma or semicolon followed by an index denotes partial or covariant differentiation, respectively.

We consider the nonstatic axially symmetric Einstein–Rosen metric

$$ds^2 = e^{2\alpha-2\beta}(dt^2 - dr^2) - r^2 e^{-2\beta} d\theta^2 - e^{2\beta} dz^2 \tag{4}$$

where α and β are functions of r and t only and r, θ, z, and t correspond to x^1, x^2, x^3, and x^4 coordinates, respectively. As a consequence of axial symmetry, we have $A_{i,\theta} = 0 = A_{i,z}$. Henceforth the lower suffixes 1 and 4 after an unknown function denote partial differentiation with respect to r and t, respectively.

Using a comoving coordinate system $[U_1 = U_2 = U_3 = 0,\ U_4 = (g^{44})^{-1/2}]$, from two of the field equations, viz., $G_{11} = -K(T_{11} + T'_{11})$ and $G_{44} = -K(T_{44} + T'_{44})$ by subtraction, we get

$$2\Lambda + \frac{K}{4\pi}(\rho - p) - \frac{K}{4\pi} g^{11} g^{44} F_{14}{}^2 + \frac{K}{4\pi} g^{22} g^{33} F_{23}{}^2 = 0 \tag{5}$$

Since the metric potentials g^{11}, g^{22}, and g^{33} are negative and g^{44} is positive and for physical distributions of interest $\Lambda \geqslant 0$ (Wesson, 1980) and $\rho \geqslant p$, the above equation implies

$$\Lambda = 0, \qquad \rho = p, \qquad F_{14} = 0, \quad \text{and} \quad F_{23} = 0 \tag{6}$$

The perfect fluid therefore reduces to the "Zel'dovich fluid" ($\rho = p$) and the cosmological constant Λ vanishes identically. Using (6) in equation (3) with $i = 4$, we get

$$\sigma = 0 \qquad (\text{since } U_4 \neq 0) \tag{7}$$

The conclusions arrived at, viz., $\Lambda = 0$, $\rho = p$, and $\sigma = 0$ are invariant statements and hold in all coordinate systems eventhough we have derived them using a comoving coordinate system. Since $F_{14} = 0 = F_{23}$, from equation (2), we get that only two components, A_2 and A_3, of electromagnetic four-potential vector A_i determine all the surviving components of F_{ij}. Putting $A_2 = \phi$ and $A_3 = \psi$, we obtain

$$F_{12} = -\phi_1, \qquad F_{13} = -\psi_1, \qquad F_{24} = \phi_4, \quad \text{and} \quad F_{34} = \psi_4 \tag{8}$$

With the help of the equations (6) and (8) and subtracting $G_{22} = -K(T_{22} +$

T'_{22}) from $G_{33} = -K(T_{33} + T'_{33})$ of the field equations (1), we have

$$\beta_{11} - \beta_{44} + \frac{\beta_1}{r} = \frac{K}{8\pi}\left[\frac{e^{2\beta}}{r^2}(\phi_1{}^2 - \phi_4{}^2) - e^{-2\beta}(\psi_1{}^2 - \psi_4{}^2)\right] \tag{9}$$

We can take equation (9) in place of the field equation $G_{33} = -K(T_{33} + T'_{33})$.

Using (6), (7), and (8), the field equations finally take the form

$$\alpha_1 = r\left[\alpha_{11} - \alpha_{44} + 2\beta_1{}^2 + \frac{K}{4\pi}\left(\frac{e^{2\beta}}{r^2}\phi_4{}^2 + e^{-2\beta}\psi_1{}^2\right)\right] \tag{10}$$

$$\alpha_4 = 2r\beta_1\beta_4 + \frac{Kr}{4\pi}\left(\frac{e^{2\beta}}{r^2}\phi_1\phi_4 + e^{-2\beta}\psi_1\psi_4\right) \tag{11}$$

$$p(=\rho) = e^{-2\alpha+2\beta}\left[\frac{2\pi}{K}\left(\alpha_{11} - \alpha_{44} - 2\beta_4{}^2 + \frac{\alpha_1}{r}\right)\right.$$

$$\left. - \frac{1}{2}\frac{e^{-2\beta}}{r^2}\phi_1{}^2 + e^{-2\beta}\psi_4{}^2\right] \tag{12}$$

$$\beta_{11} - \beta_{44} + \frac{\beta_1}{r} = \frac{K}{8\pi}\left[\frac{e^{2\beta}}{r^2}(\phi_1{}^2 - \phi_4{}^2) - e^{-2\beta}(\psi_1{}^2 - \psi_4{}^2)\right] \tag{13}$$

$$\phi_1\psi_1 - \phi_4\psi_4 = 0 \tag{14}$$

$$\phi_{11} - \phi_{44} - \frac{\phi_1}{r} = 2\beta_4\phi_4 - 2\beta_1\phi_1 \tag{15}$$

$$\psi_{11} - \psi_{44} + \frac{\psi_1}{r} = 2\beta_1\psi_1 - 2\beta_4\psi_4 \tag{16}$$

The problem now is to solve the overdetermined set of highly nonlinear differential equations (10) to (16) for the unknowns ϕ, ψ, α, β, and p $(=\rho)$. The satisfaction of the field equations by direct substitution of the solutions obtained by us meets the requirement of the overdeterminacy.

3. SOLUTIONS

Because of the complicated structure of the field equations, we solve them under the following three cases:

Case III A: $\phi = 0$, $\psi \neq 0$ (i.e., $F_{13} \neq 0 \neq F_{34}$, the rest of $F_{ij} = 0$);
Case III B: $\phi = 0$, $\psi = 0$ (i.e., $F_{12} \neq 0 \neq F_{24}$, the rest of $F_{ij} = 0$); and
Case III C: $\phi \neq 0$, $\psi \neq 0$ (i.e., $F_{14} = 0 = F_{23}$, the rest of $F_{ij} \neq 0$).

Case III A: $\phi=0$, $\psi\neq 0$ (i.e., $F_{13}\neq 0\neq F_{34}$, Rest of $F_{ij}=0$). For this case the equations (14) and (15) are satisfied identically (since $\psi=0$) and thus we are required to solve the field equations (10)–(13) and (15) for the unknowns ψ, α, β, and $p(=\rho)$.

Case III A (1). Let us assume ψ to be a function of r only. Equation (16) reduces to

$$\beta=\tfrac{1}{2}\left[\log(r\psi_1)+L(t)\right] \tag{17}$$

where L is arbitrary function of time only. Substituting this value of β from (17) in (13), we get

$$\frac{d^2}{dr^2}\log(r\psi_1)+\frac{1}{r}\frac{d}{dr}\log(r\psi_1)-\frac{d^2L}{dt^2}=-\frac{K}{4\pi}\frac{\psi_1}{r}e^{-L} \tag{18}$$

As ψ is a function of r and L is a function of t only, equation (18) will hold if either ψ_1/r is a constant or L is a constant.

Subcase III A(1) a. Assuming L to be a constant, say

$$L=-\log f \tag{19}$$

we get from (18)

$$\frac{d^2}{dr^2}\log(\psi_1 r)+\frac{1}{r}\frac{d}{dr}\log(\psi_1 r)=-u\frac{\psi_1}{r} \tag{20}$$

where we take $u=Kf/4\pi$. Solving equation (20), we get

$$\psi=b-\frac{q}{u}\tanh\left(-\frac{q}{2}\log r+s\right) \tag{21}$$

Here and in what follows small Latin letters except r, t, and z denote arbitrary constants unless otherwise stated.

Substituting (19) and (21) in (17), we get

$$\beta=\frac{1}{2}\log\left[\frac{q^2}{2uf}\operatorname{sech}^2\left(-\frac{q}{2}\log r+s\right)\right] \tag{22}$$

Solving the equations (10) and (11) with the help of (21) and (22), we obtain

$$\alpha=\frac{q^2}{4}\log r+hr^2+g \tag{23}$$

Putting the values of ψ, β, and α from (21)–(23) in (12) we have the pressure as

$$p(=\rho)=\frac{q^2h}{u^2}r^{\frac{-q^2}{2}}\operatorname{sech}^2\left(-\frac{q}{2}\log r+s\right)e^{-2(hr^2+g)} \tag{24}$$

The solution of the field equations is given by (21)–(24).

Subcase IIIA(1)b. In the alternative case considering ψ_1/r to be a constant, say f, we get

$$\psi=\tfrac{1}{2}fr^2+d \tag{25}$$

With this value of ψ equation (18) reduces to

$$L_{44}-eu^{-L}=0 \tag{26}$$

The general solution of (26) is

$$L=\log\left[\frac{2u}{q^2}\cosh^2\left(\frac{q}{2}t+s\right)\right] \tag{27}$$

Substituting the values of ψ and L from (25) and (27) in (17), we get

$$\beta=\frac{1}{2}\log\left[\frac{2uf}{q^2}r^2\cosh^2\left(\frac{q}{2}t+s\right)\right] \tag{28}$$

Again substituting the values of ψ and β from (25) and (28) in (11) and subsequently in (10) and solving, we obtain

$$\alpha=\log\left[r\cosh^2\left(\frac{q}{2}t+s\right)\right]+hr^2+g \tag{29}$$

Finally putting the values of ψ, β, and α from (25), (28), and (29) in (12), we get

$$p(=\rho)=\frac{f^2}{2q^2}(8h-q^2)\operatorname{sech}^2\left(\frac{q}{2}t+s\right)e^{-2(hr^2+g)} \tag{30}$$

Hence in this case (25), (28), (29), and (30) constitute the solution of the field equations.

Case III A(2). In this case let us assume ψ to be a function of time only. Thus we get from (16)

$$\beta=\tfrac{1}{2}\left[\log\psi_4-L(r)\right] \tag{31}$$

where L is an arbitrary function of r only.

Substituting (31) in (13), we get

$$\frac{d^2L}{dr^2}+\frac{1}{r}\frac{dL}{dr}+\frac{d^2}{dt^2}\log\psi_4=-\frac{K}{4\pi}\psi_4 e^L \tag{32}$$

Since ψ is a function of time only and L is a function of r only, the equation (32) will hold if and only if either L or ψ_4 is constant.

Subcase III A(2) a. First assuming L to be a constant, say,

$$L=\log f \tag{33}$$

and solving equation (32), we get

$$\psi=b-\frac{q}{u}\tanh\left(-\frac{q}{2}t+s\right) \tag{34}$$

Putting the values of L and ψ from (33) and (34) in (31), we get

$$\beta=\frac{1}{2}\log\left[\frac{q^2}{2uf}\operatorname{sech}^2\left(-\frac{q}{2}t+s\right)\right] \tag{35}$$

Substituting the values of ψ and β from (34) and (35) in (10) and (11), and solving we obtain

$$\alpha=hr^2+g \tag{36}$$

Finally, inserting the values of ψ, β, and α from (34)–(36) in (12), we get the pressure

$$p(=\rho)=\frac{q^2}{8u^2}(8h-q^2)\operatorname{sech}^2\left(-\frac{q}{2}t+s\right)e^{-2(hr^2+g)} \tag{37}$$

Thus in this case the solution is given by (34)–(37).

Subcase III A(2) b. In the case when ψ_4 is constant, say f, we write

$$\psi=ft+d \tag{38}$$

With this value of ψ, equation (32) yields the solution

$$L = -\log\left[\frac{2u}{q^2} r^2 \cosh^2\left(\frac{q}{2}\log r + s\right)\right] \tag{39}$$

With the help of (31), (38) and (39), we get

$$\beta = \frac{1}{2}\log\left[\frac{2uf}{q^2} r^2 \cosh^2\left(\frac{q}{2}\log r + s\right)\right] \tag{40}$$

These values of ψ and β when substituted in (11) give, after integration,

$$\alpha = S(r) \tag{41}$$

where S is an arbitrary function of r only. Again substituting ψ, β, and α from (38), (40), and (41) in (10) we get the differential equation

$$r^2 S_{11} - rS_1 = -2 - \frac{q^2}{2}\tanh^2\left(\frac{q}{2}\log r + s\right) - 2q\tanh\left(\frac{q}{2}\log r + s\right) \tag{42}$$

It is difficult to get the particular integral of the above differential equation for general value of q. Hence we solve for a certain value of q, say for $q = 2$. The solution of (42) for this case is

$$\alpha = \log\left[r^2\cosh^2(\log r + s)\right] + hr^2 + g \tag{43}$$

Now putting the values of ψ, β, and α (when $q = 2$) in (12), we get

$$p(=\rho) = f^2 hr^{-2}\operatorname{sech}^2(\log r + s)e^{-2(hr^2+q)} \tag{44}$$

Thus in this case the solution of the field equations is given by

$$\psi = ft + d, \qquad \beta = \frac{1}{2}\log\left[\frac{uf}{2} r^2\cosh^2(\log r + s)\right]$$

$$\alpha = \log\left[r^2\cosh^2(\log r + s)\right] + hr^2 + g \tag{45}$$

$$p(=\rho) = f^2 hr^{-2}\operatorname{sech}^2(\log r + s)e^{-2(hr^2+g)}$$

Case IIIB: $\phi \neq 0$, $\psi = 0$ (i.e., $F_{13} \neq 0 \neq F_{34}$ and Rest of $F_{ij} = 0$). Since $\psi = 0$, equations (14) and (16) are satisfied identically and hence we have to solve the rest of the field equations (10)–(13) and (15) for the unknowns ϕ, α, β, and $p(=\rho)$.

Case III B(1). The following two sets of solutions are obtained for the case when ϕ is a function of r only [similar to the case A(1)]:

Subcase III B(1) a.

$$\phi = b + \frac{2}{u}\tanh(\log r + s), \qquad \beta = \frac{1}{2}\log\left[\frac{uf}{2} r^2 \cosh^2(\log r + s)\right]$$

$$\alpha = \log\left[r\cosh^2(\log r + s)\right] + hr^2 + g \tag{46}$$

$$p(=\rho) = hf^2 r^{-2}\operatorname{sech}^2(\log r + s)e^{-2(hr^2+g)}$$

Subcase III B(1) b

$$\phi = \frac{1}{2}fr^2 + d, \qquad \beta = \frac{1}{2}\log\left[\frac{q^2}{2uf}\operatorname{sech}^2\left(-\frac{q}{2}t + s\right)\right]$$

$$\alpha = hr^2 + g \tag{47}$$

$$p(=\rho) = \frac{q^2}{8u^2}(8h - q^2)\operatorname{sech}^2\left(-\frac{q}{2}t + s\right)\exp\left[-2(hr^2 + g)\right]$$

Case III B(2). The following two sets of solutions are obtained for the case when ϕ is a function of time only [similar to the case A(2)]:

Subcase III B(2) a.

$$\phi = b - \frac{q}{u}\tanh\left(-\frac{q}{2}t + s\right), \qquad \beta = \frac{1}{2}\log\left[\frac{2uf}{q^2} r^2 \cosh^2\left(-\frac{q}{2}t + s\right)\right]$$

$$\alpha = \log\left[r\cosh^2\left(-\frac{q}{2}t + s\right)\right] + hr^2 + g \tag{48}$$

$$p(=\rho) = \frac{r^2}{q^2}\left(4h - \frac{q^2}{2}\right)\operatorname{sech}^2\left(-\frac{q}{2}t + s\right)e^{-2(hr^2+g)}$$

Subcase III B(2) b.

$$\phi = ft + d, \qquad \beta = \frac{1}{2}\log\left[\frac{q^2}{2uf}\operatorname{sech}^2\left(-\frac{q}{2}\log r + s\right)\right]$$

$$\alpha = \frac{q^2}{4}\log r + hr^2 + g \tag{49}$$

$$p(=\rho) = \frac{q^2 h}{u^2} r^{-q^2/2}\operatorname{sech}^2\left(-\frac{q}{2}\log r + s\right)e^{-2(hr^2+g)}$$

Case III C: $\phi = 0$, $\psi \neq 0$ (i.e., $F_{14} = 0 = F_{23}$, Rest of $F_{ij} \neq 0$). In this case the equation (14) implies the following three possibilities or else we find the cases already discussed (viz., cases A and B):

Subcase IIIC(1): ϕ and ψ are functions of the type $\phi = \psi = F(r+t)$ or $\phi = \psi = F(r-t)$, where F is an arbitrary function of $(r+t)$ or $(r-t)$;

Subcase IIIC(2): ϕ is a function r and ψ is a function of t only; and

Subcase IIIC(3): ϕ is a function of time and ψ is a function of r only.

Subcase IIIC(1). As it is impossible to get the integral of an arbitrary function we consider the particular functional forms for ϕ and ψ as

$$\phi = \psi = (r-t)^n \tag{50}$$

where n is any real number except zero. Then both the equations (15) and (16) lead to a single equation, viz.,

$$\beta_1 + \beta_4 = \frac{1}{2r} \tag{51}$$

With the help of (50) equation (13) reduces to

$$\beta_{11} - \beta_{44} + \frac{\beta_1}{r} = 0 \tag{52}$$

Solution of equations (51) and (52) is

$$\beta = \tfrac{1}{2}\log(r/m) \tag{53}$$

Solving equations (10) and (11) with the help of (50) and (53), we obtain

$$\alpha = \frac{K(1+m^2)n^2}{4\pi m(2n-1)}(r-t)^{2n-1} + \frac{1}{4}\log r + hr^2 + g \tag{54}$$

Finally putting these values of ϕ, ψ, β, and α from (50), (53), and (54) in (12), we get the pressure as

$$p(=\rho) = \frac{8\pi h}{Km} r^{1/2} \exp\left\{-2\left[\frac{K(1+m^2)n^2}{4\pi m(2n-1)}(r-t)^{2n-1} + hr^2 + g\right]\right\} \tag{55}$$

In this case (50) and (53)–(55) constitute the solution of the field equations.

Subcase IIIC(2). Since $\phi_4 = 0$ and $\psi_1 = 0$ for this case, equation (15) leads to $\beta = \frac{1}{2}[\log(r/\phi_1) + N(t)]$ and equation (16) becomes $\beta = \frac{1}{2}[\log \psi_4 + M(r)]$, where M and N are arbitrary functions of r and t, respectively. Comparing these values of β, we get

$$\beta = \tfrac{1}{2} \log(r\psi_4/\phi_1) \tag{56}$$

Substituting β from (56) in (13), we obtain

$$\frac{d^2}{dr^2} \log(\phi_1/r) + \frac{1}{r}\frac{d}{dr} \log(\phi_1/r) + \frac{d^2}{dt^2} \log \psi_4 = -\frac{K}{2\pi}\frac{\phi_1\psi_4}{r} \tag{57}$$

As before since ϕ is a function of r and ψ is a function of t only, equation (57) will hold only if (ϕ_1/r) is a constant or ψ_4 is a constant.

Subcase IIIC(2)a. Assuming (ϕ_1/r) to be a constant, say f, we get

$$\phi = \tfrac{1}{2} f r^2 + d \tag{58}$$

With this value of ϕ, the general solution of equation (57) is

$$\psi = b - \frac{q}{2u} \tanh\left(-\frac{q}{2}t + s\right) \tag{59}$$

Now from equations (56), (58), and (59), we have

$$\beta = \frac{1}{2} \log\left[\frac{q^2}{4uf} \operatorname{sech}^2\left(-\frac{q}{2}t + s\right)\right] \tag{60}$$

Successively solving equations (11) and (10) with the help of ϕ, ψ, and β from (58)–(60), we obtain

$$\alpha = hr^2 + g \tag{61}$$

Finally substituting ϕ, ψ, β, and α from (58)–(61) in (12), we get pressure as

$$p(=\rho) = \frac{q^2}{8u^2}\left(4h - \frac{q^2}{2}\right) \operatorname{sech}^2\left(-\frac{q}{2}t + s\right) \exp[-2(hr^2 + g)] \tag{62}$$

Subcase IIIC(2)b. In the alternative case when ψ_4 is a constant we get the following solutions:

$$\psi = ft + d, \qquad \phi = b + \frac{1}{u}\tanh(\log r + s)$$

$$\beta = \tfrac{1}{2}\log\left[ufr^2\cosh^2(\log r + s)\right]$$

$$\alpha = \log\left[r^2\cosh^2(\log r + s)\right] + hr^2 + g$$

$$p(=\rho) = 2hf^2r^{-2}\operatorname{sech}^2(\log r + s)e^{-2(hr^2+g)} \tag{63}$$

Subcase IIIC(3). Adopting the procedure of the subcase III C(2), the following two sets, IIIC(3)a and IIIC(3)b, are obtained:

Subcase IIIC(3)a.

$$\psi = \frac{1}{2}fr^2 + d, \qquad \phi = b - \frac{q}{2u}\tan h\left(-\frac{q}{2}t + s\right)$$

$$\beta = \frac{1}{2}\log\left[\frac{4uf}{q^2}r^2\cosh^2\left(-\frac{q}{2}t + s\right)\right]$$

$$\alpha = \log\left[r\cosh^2\left(-\frac{q}{2}t + s\right)\right] + hr^2 + g$$

$$p(=\rho) = \frac{f^2}{q^2}(sh - q^2)\operatorname{sech}^2\left(-\frac{q}{2}t + s\right)\exp\left[-2(hr^2 + g)\right] \tag{64}$$

Subcase IIIC(3)b.

$$\phi = ft + d, \qquad \psi = b - \frac{q}{2u}\tanh\left(-\frac{q}{2}\log r + s\right)$$

$$\beta = \frac{1}{2}\log\left[\frac{q^2}{4uf}\operatorname{sech}^2\left(-\frac{q}{2}\log r + s\right)\right]$$

$$\alpha = \frac{q^2}{4}\log r + hr^2 + g$$

$$p(=\rho) = \frac{hq^2}{2u^2}r^{-q^2/2}\operatorname{sech}^2\left(-\frac{q}{2}\log r + s\right)e^{-2(hr^2+g)} \tag{65}$$

4. CYLINDRICALLY SYMMETRIC BD–MAXWELL SOLUTIONS

In this section following Tabensky and Taub (1973) and Rao et al. (1974a, b) we have generated the solutions of the BD theory from those of the Einstein theory. The transformation reads as follows:

$$p = \rho = g^{ij} V_i V_j, \tag{66}$$

$$\Phi = \exp\left[(2V)^{1/2}/(w+3/2)^{1/2}\right] \tag{67}$$

$$\Phi g'_{ij} = g_{ij} \tag{68}$$

where g'_{ij}, Φ are the solutions of the BD theory with the coupling constant $w \neq -3/2$ and g_{ij} are the solutions of Einstein's theory. Relation (66) yields a zero-mass scalar V, and relation (67) constructs the scalar Φ of BD theory. Thus BD Maxwell solutions can be generated from Einstein–Maxwell–Zel'dovich solutions where the electromagnetic field remains unchanged by the above transformations. That is,

$$E_{ij} = F_{ij} \tag{69}$$

where E_{ij} is the electromagnetic field tensor of the BD–Maxwell field and F_{ij} is that of the Einstein–Maxwell field.

Let us consider the solution (47) in which $p(=\rho)$ is obtained from equation (12) by using β and α from (42) and (45) for the case $q=2$. Thus

$$p(=\rho) = \frac{8\pi h}{K} e^{-2\alpha+2\beta} \tag{70}$$

Again for the metric (4), equation (66) reduces to

$$p(=\rho) = e^{-2\alpha+2\beta}\left(V_4^2 - V_1^2\right) \tag{71}$$

Equations (70) and (71) together lead to

$$V_4^2 - V_1^2 = \frac{8\pi h}{K} \tag{72}$$

It is to be noted here that the BD scalar Φ corresponding to the zero-mass scalar V is not only governed by the Equation (72) but also by the scalar wave equation [for the metric (4)]

$$g^{ij}V_{;ij}=0 \tag{73}$$

which is a consequence of the restriction on Zel'dovich fluid to be a zero-mass field. Hence a solution satisfying (72) and (73) is given by

$$V=\left(\frac{8\pi h}{K}\right)^{1/2} t + a \tag{74}$$

The solutions for the BD–Maxwell field equations

$$G_{ij} \equiv R_{ij} - \frac{1}{2}g'_{ij}R = -\frac{K}{4\pi\Phi}\left[-E_{is}E_j^s + \frac{1}{4}g'_{ij}E_{ab}E^{ab}\right]$$
$$-\frac{w}{\Phi^2}\left[\Phi_{,i}\Phi_{,j} - \frac{1}{2}g'_{ij}\Phi'^{k}\Phi_{,k}\right] - \frac{1}{\Phi}\left[\Phi_{;ij} - g'_{ij}\Phi'^{k}_{;k}\right]$$

$$E^{ij}_{;j}=0 \tag{75}$$

$$E_{ij}=A'_{i,j}-A'_{j,i} \qquad \text{(surviving components of } E_{ij} \text{ are determined by } A'_2=\zeta \text{ and } A'_3=\eta) \tag{76}$$

and

$$\Phi^{k}_{;k}=0 \qquad \text{(with coupling constant } w \neq -3/2) \tag{77}$$

for the space-time described by the general cylindrically symmetric metric (Marder, 1958a, b)

$$ds^2 = e^{2\bar{\alpha}-2\bar{\beta}}(dt^2-dr^2) - r^2e^{-2\bar{\beta}}d\theta^2 - e^{2\bar{\beta}+2\bar{\nu}}dr^2 \tag{78}$$

can be generated easily with the help of (67), (68), and (74) as

$$\zeta=0, \qquad \eta=ft+d, \qquad \bar{\alpha}=2\log[r\cosh(\log r+s)]+hr^2+g$$

$$\bar{\beta}=\frac{1}{2}\log\left[\frac{uf}{2}r^2\cosh^2(\log r+s)\right]+\frac{(8\pi h/K)^{1/2}t+a}{(2w+3)^{1/2}} \tag{79}$$

$$\bar{\nu}=-2\frac{(8\pi h/K)^{1/2}t+a}{(2w+3)^{1/2}}=-\log\Phi$$

It may be verified by direct substitution that (79) is the solution of (75)–(78) for $K = 4\pi$.

5. CONCLUSIONS

It may be mentioned here that the solutions of the purely Einstein–Maxwell source-free fields (i.e., $p = \rho = 0$) correspond to those obtained by Misra and Radhakrishna (1962) with proper identification of arbitrary constants. Similarly it can be shown that our solutions without the contribution of the Zel'dovich fluid correspond to those obtained by Rao et al. (1972) (with scalar field $V = 0$ and adjusting the arbitrary constants involved).

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